A NOTE ON LINEAR CODES AND NONASSOCIATIVE ALGEBRAS OBTAINED FROM SKEW-POLYNOMIAL RINGS

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ABSTRACT. Different approaches to construct linear codes using skew polynomials can be unified by using the nonassociative algebras built from skew-polynomial rings by Petit.

INTRODUCTION

In recent years, several classes of linear codes were obtained from skew-polynomial rings (also called Ore rings). Using this approach, self-dual codes with a better minimal distance for certain lengths than previously known were constructed: while the classical *cyclic codes* of length m over a finite field $\mathbb{F}_q$ are obtained from ideals in the commutative ring $\mathbb{F}_q[t]/(t^m-1)$, and *constacyclic codes* from ideals in the commutative ring $\mathbb{F}_q[t]/(t^m-d)$, $d\in\mathbb{F}_q$, *ideal σ-codes* are associated with left ideals in the non-commutative ring $\mathbb{F}_q[t;\sigma]/(t^m-1)$, where $t^m-1\in R$ is a two-sided element in the twisted polynomial ring $\mathbb{F}_q[t;\sigma]$, $\sigma\in\mathrm{Aut}(\mathbb{F}_q)$, see [4]. Because t^m-1 is required to be a two-sided element in order for $\mathbb{F}_q[t;\sigma]/(t^m-1)$ to be a ring, this enforces restrictions on the possible lengths of the codes obtained: t^m-1 is two-sided if and only if the order n of σ divides m [10, (15)].

If Rf denotes the left ideal generated by an element $f\in R$, R a ring, then R/Rf is a left R-module. In [3], linear codes associated with left R-submodules Rg/Rf of R/Rf are considered, where $R=\mathbb{F}_q[t;\sigma]$ and g is a right divisor of f. These codes are called *module σ-codes*. Another generalization is discussed in [1] and [6], where codes obtained from submodules of the R-module R/Rf for some monic $f\in R$ are investigated, where now $R=\mathbb{F}_q[t;\sigma,\delta]$ is a skew-polynomial ring.

We show that all these approaches can be unified since the codes mentioned above are associated to the left ideals of the nonassociative algebra S_f defined by Petit [10]. For a unital division ring D (which here will be a finite field), and a polynomial f in the skew-polynomial ring $R=D[t;\sigma,\delta]$, Petit defined a nonassociative ring on the set $R_m=\{h\in D[t;\sigma,\delta]\,|\,\deg(h)<m\}$, using right division $g\circ h=gh \bmod_r f$ to define the algebra multiplication. $S_f=(R_m,\circ)$ is a nonassociative algebra over $F_0=\{a\in D\,|\,ah=ha \text{ for all } h\in S_f\}$ whose left ideals are generated by the polynomials g which are right divisors of f.

The scenarios treated with respect to the linear codes mentioned above all require f to be reducible, so the corresponding, not necessarily associative, algebra S_f is not allowed to be a division algebra here. The cyclic submodules studied in [1], [2] are exactly the left ideals

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in the algebra S_f. The (σ,δ)-codes of [6] are the codes $\mathcal{C}$ associated to a left ideal of S_f generated by a right divisor g of f with $f \in K[t;\sigma,\delta]$. We show that if σ is an automorphism of $K=\mathbb{F}_q$ and $\mathcal{C}$ a linear code over $\mathbb{F}_q$ of length m, then $\mathcal{C}$ is a σ-constacyclic code with constant d iff the skew-polynomial representation $\mathcal{C}(t)$ with elements $a(t)$ obtained from $(a_0,\dots,a_{m-1})\in\mathcal{C}$ is a left ideal of S_f with $f=t^m-d\in R=\mathbb{F}_q[t;\sigma]$, generated by a monic right divisor g of f in R.

1. Preliminaries

1.1. **Nonassociative algebras.** Let F be a field and let A be a finite-dimensional F-vector space. We call A an *algebra* over F if there exists an F-bilinear map $A\times A\to A$, $(x,y)\mapsto x\cdot y$, denoted simply by juxtaposition xy, the *multiplication* of A. An algebra A is called *unital* if there is an element in A, denoted by 1, such that $1x=x1=x$ for all $x\in A$. We will only consider unital algebras.

An algebra $A\neq 0$ is called a *division algebra* if for any $a\in A$, $a\neq 0$, the left multiplication with a, $L_a(x)=ax$, and the right multiplication with a, $R_a(x)=xa$, are bijective. A is a division algebra if and only if A has no zero divisors ([12], pp. 15, 16).

1.2. **Skew-polynomial rings.** In the following, we use results by Jacobson [7] and Petit [10]. Let D be a unital associative division ring, σ a ring endomorphism of D and δ a *left σ-derivation* of D, i.e. an additive map such that

$$\delta(ab)=\sigma(a)\delta(b)+\delta(a)b$$

for all $a,b\in D$, implying $\delta(1)=0$. The *skew-polynomial ring* $D[t;\sigma,\delta]$ is the set of polynomials

$$a_0+a_1t+\dots+a_nt^n$$

with $a_i\in D$, where addition is defined term-wise and multiplication by

$$ta=\sigma(a)t+\delta(a) \quad (a\in D).$$

$D[t;\sigma]=D[t;\sigma,0]$ is called a *twisted polynomial ring* and $D[t;\delta]=D[t;id,\delta]$ a *differential polynomial ring*. For the special case that $\sigma=id$ and $\delta=0$, we obtain the usual ring of left polynomials $D[t]=D[t;id,0]$, often also denoted $D_L[t]$ in the literature, with its multiplication given by

$$(\sum_{i=1}^{s}a_it^i)(\sum_{i=1}^{t}b_it^i)=\sum_{i,j}a_ib_jt^{i+j}.$$

If D has finite dimension over its center and σ is a ring automorphism of D, then $R=D[t;\sigma,\delta]$ is either a twisted polynomial or a differential polynomial ring by a linear change of variables [7, Thm. 1.2.21]. Note also that if σ and δ are F-linear maps then $D[t;\sigma,\delta]\cong D[t]$ by a linear change of variables.

For $f=a_0+a_1t+\dots+a_nt^n$ with $a_n\neq 0$ define $\deg(f)=n$ and $\deg(0)=-\infty$. Then $\deg(fg)=\deg(f)+\deg(g)$. An element $f\in R$ is *irreducible* in R if it is no unit and it has no proper factors, i.e if there do not exist $g,h\in R$ with $\deg(g),\deg(h)<\deg(f)$ such that $f=gh$.

$R = D[t;\sigma,\delta]$ is a left principal ideal domain (i.e., every left ideal in R is of the form Rf) and there is a right-division algorithm in R [7, p. 3]: for all $g, f \in R$, $g \neq 0$, there exist unique $r, q \in R$, and $\deg(r) < \deg(f)$, such that

$$g = qf + r.$$

(We point out that our terminology is the one used by Petit [10], Lavrauw and Sheekey [9], and in the coding literature; it is different from Jacobson's [7], who calls what we call right a left division algorithm and vice versa.)

1.3. **How to obtain nonassociative division algebras from skew-polynomial rings.** Let D be a unital associative division algebra, σ an injective ring homomorphism and $f \in D[t;\sigma,\delta]$ of degree m.

Definition 1. (cf. [10, (7)]) Let $\mathrm{mod}_r f$ denote the remainder of right division by f. Then

$$R_m = \{g \in D[t;\sigma,\delta] \,|\, \deg(g) < m\}$$

together with the multiplication

$$g \circ h = gh \;\mathrm{mod}_r f$$

becomes a unital nonassociative algebra $S_f = (R_m, \circ)$ over

$$F_0 = \{a \in D \,|\, ah = ha \text{ for all } h \in S_f\}.$$

This algebra is also denoted by R/Rf [10, 11] if we want to make clear which ring R is involved in the construction. F_0 is a subfield of D [10, (7)].

Remark 1. For $f(t) = t^m - d \in R = D[t;\sigma]$, the multiplication in S_f is defined via

$$(at^i)(bt^j) = \begin{cases} a\sigma^i(b)t^{i+j} & \text{if } i+j < m, \\ a\sigma^i(b)t^{(i+j)-m}d & \text{if } i+j \geq m, \end{cases}$$

for all $a, b \in D$ and then linearly extended.

2. Linear codes associated to left ideals of S_f

Let K be a finite field, σ an automorphism of K and $F = \mathrm{Fix}(\sigma)$, $[K : F] = n$. By a linear base change we could always assume $\delta = 0$. However, [1] and [6] show that this limits the choices of available codes.

Unless specified otherwise, let $R = K[t;\sigma,\delta]$ and $f \in R$ be a monic polynomial of degree m. Analogously as for instance in [1], [2], [3], [4], we associate to an element $a(t) = \sum_{i=0}^{m-1} a_i t^i$ in S_f the vector $(a_0, \ldots, a_{m-1})$. Our codes $\mathcal{C}$ of length m consist of all $(a_0, \ldots, a_{m-1})$ obtained this way from the elements $a(t) = \sum_{i=0}^{m-1} a_i t^i$ in a left ideal I of S_f. Conversely, for a linear code $\mathcal{C}$ of length n we denote by $\mathcal{C}(t)$ the set of skew-polynomials $a(t) = \sum_{i=0}^{m-1} a_i t^i \in S_f$ associated to the codewords $(a_0, \ldots, a_n) \in \mathcal{C}$.

Proposition 2. *Let D be a unital associative division ring and $f \in R = D[t;\sigma,\delta]$.*
(i) All left ideals in S_f are generated by some monic right divisor g of f in R.
(ii) If f is irreducible, then S_f has no non-trivial left ideals.

Proof. (i) The proof is analogous to the one of [5, Lemma 1], only that now we are working in the nonassociative ring S_f: Let I be a left ideal of S_f. If $I = \{0\}$ then $I = (0)$. So suppose $I \neq (0)$ and choose a monic non-zero polynomial g in $I \subset R_m$ of minimal degree. For $p \in I \subset R_m$, a right division by g yields unique $r, q \in R$ with $\deg(r) < \deg(g)$ such that

$$p = qg + r$$

and hence $r = p - qg \in I$. Since we chose $g \in I$ to have minimal degree, we conclude that $r = 0$, implying $p = qg$ and so $I = Rg$.
(ii) follows from (i). □

Corollary 3. *(i) The cyclic submodules studied in* [1], [2] *are exactly the left ideals in the algebra S_f where $f \in \mathbb{F}_q[t; \sigma, \delta]$.*
(ii) The (σ, δ)-codes $\mathcal{C}$ in [6] *are exactly the codes associated to a left ideal of S_f generated by a non-trivial right divisor g of $f \in D[t; \sigma, \delta]$, whenever D is an associative division ring (usually, $D = \mathbb{F}_q$).*

Note that when we look at the nonassociative case, where f is not two-sided anymore, it can happen that f is irreducible in $K[t; \sigma, \delta]$, hence does not have any non-trivial right divisors g.

Remark 4. Let $m \geq 2$. Since for $a(t) \in S_f$ also $ta(t) \in S_f$, we obtain for $f(t) = t^m - d \in K[t; \sigma]$ that

$$ta(t) = \sigma(a_0)t + \sigma(a_1)t^2 + \cdots + \sigma(a_{m-1})t^m = \sigma(a_{m-1})d + \sigma(a_0)t + \sigma(a_1)t^2 + \cdots + \sigma(a_{m-2})t^{m-1}$$

in S_f, so that

$$(a_0, \ldots, a_{m-1}) \in \mathcal{C} \Rightarrow (\sigma(a_{m-1})d, \sigma(a_0), \ldots, \sigma(a_{m-2})) \in \mathcal{C}$$

is a σ-constacyclic code (even if S_f is division). With the same argument, every left ideal Rg in S_f with $g \in R$ a right divisor of $f = t^m - d$ yields a σ-constacyclic code $\mathcal{C}$ for $d \neq 1$ and a σ-cyclic code for $d = 1$.

In [5, Theorem 1] it is shown that the code words of a σ-cyclic code are coefficient tuples of elements $a(t) = \sum_{i=0}^{m-1} a_i t^i \in \mathbb{F}_q[t; \sigma]/(t^m - 1)$, which are left multiples of some element $g \in \mathbb{F}_q[t; \sigma]/(t^m - 1)$ which is a right divisor of f, under the assumption that the order n of σ divides m. The assumption that n divides m guarantees that Rf is a two-sided ideal, i.e. that S_f is associative, but is not required:

Theorem 5. *Let σ be an automorphism of $K = \mathbb{F}_q$ and $\mathcal{C}$ a linear code over $\mathbb{F}_q$ of length m. Then $\mathcal{C}$ is a σ-constacyclic code (with constant d) iff the skew-polynomial representation $\mathcal{C}(t)$ with elements $a(t)$ obtained from $(a_0, \ldots, a_{m-1}) \in \mathcal{C}$ is a left ideal of S_f with $f = t^m - d \in R = \mathbb{F}_q[t; \sigma]$, generated by a monic right divisor g of f in R.*

Proof. $\Leftarrow$: This is Remark 8.

$\Rightarrow$: The argument is analogous to the proof of [5, Theorem 1]: If we have a σ-constacyclic code $\mathcal{C}$, then its elements define polynomials $a(t) \in \mathbb{F}_q[t; \sigma] = K[t; \sigma]$. These form a left

ideal $\mathcal{C}(t)$ of S_f with $f = t^m - d \in \mathbb{F}_q[t;\sigma]$: The code is linear and so the skew-polynomial representation $\mathcal{C}(t)$ is an additive group. For $(a_0, \ldots, a_{m-1}) \in \mathcal{C}$,

$$ta(t) = \sigma(a_0)t + \sigma(a_1)t^2 + \cdots + \sigma(a_{m-1})t^m$$

and since $f = t^m - d$ we get in $S_f = R/Rf$ that

$$ta(t) = \sigma(a_{m-1})d + \sigma(a_0)t + \sigma(a_1)t^2 + \cdots + \sigma(a_{m-2})t^{m-1}.$$

Since $\mathcal{C}$ is σ-constacyclic with constant d, $ta(t) \in \mathcal{C}(t)$. Clearly, by iterating this argument, also $t^s a(t) \in \mathcal{C}(t)$ for all $s \le m-1$. By iteration and linearity of $\mathcal{C}$, thus $h(t)a(t) \in \mathcal{C}(t)$ for all $h(t) \in R_m$, so $\mathcal{C}(t)$ is closed under multiplication and a left ideal of S_f. □

Corollary 6. *Let σ be an automorphism of $K = \mathbb{F}_q$ and $\mathcal{C}$ a linear code over $\mathbb{F}_q$ of length m. Then $\mathcal{C}$ is a σ-cyclic code iff the skew-polynomial representation $\mathcal{C}(t)$ with elements $a(t)$ obtained from $(a_0, \ldots, a_{m-1}) \in \mathcal{C}$ is a left ideal of S_f generated by a monic right divisor g of $f = t^m - 1 \in R = \mathbb{F}_q[t;\sigma]$.*

Remark 7. Let $f(t) = t^m - d \in R = K[t;\sigma]$ and $F = \mathrm{Fix}(\sigma)$. Then f is a two-sided element (thus S_f associative and f reducible) iff m divides the order n of σ and $d \in F$ by [10, (7), (9)]. For $d = 1$ in particular, f is two-sided iff m divides the order n of σ. Any right divisor g of degree k of, for instance, $f = t^m - d$ can be used to construct a σ-constacyclic $[m, m-k]$-code (with constant d). (If f is not two-sided, it can happen that f is irreducible in $K[t;\sigma]$, hence does not have any non-trivial right divisors g.) We note:
(i) $f(t) = t^3 - d$ is reducible in R if and only if

$$\sigma(z)^2\sigma(z)z = d \text{ or } \sigma(z)^2\sigma(z)z = d$$

for some $z \in K$ [10, (18)] . (Thus $t^3 - 1$ is always reducible in $K[t;\sigma]$.)
(ii) Suppose m is prime and F contains a primitive mth root of unity. Then $f(t) = t^m - d$ is reducible in R if and only if

$$d = \sigma^{m-1}(z)\cdots\sigma(z)z \text{ or } \sigma^{m-1}(d) = \sigma^{m-1}(z)\cdots\sigma(z)z$$

for some $z \in K$ [10, (19)]. (Thus $t^m - 1$ is always reducible in $K[t;\sigma]$, if F contains a primitive mth root of unity.)
(iii) Let K/F have degree m, $\mathrm{Gal}(K/F) = \langle\sigma\rangle$ and $R = K[t;\sigma]$, $f = t^m - d$ with $d \notin F$. Then the nonassociative cyclic algebra $(K/F, \sigma, d)$ is the algebra S_f with $R = K[t;\sigma^{-1}]$ and $f(t) = t^m - d$ (cf. [10, p. 13-13]).
(a) If the elements $1, d, \ldots, d^m$ are linearly dependent over F, then f is reducible.
(b) If m is prime then f is irreducible [13] and thus there are no σ-constacyclic codes with constant d apart from the $[m, m]$-code associated with S_f itself.

When working over finite fields, the division algebras S_f are finite semifields which are closely related to the semifields constructed by Johnson and Jha [8] obtained by employing semi-linear transformations. Results for these semifields and their spreads might be useful for future linear code constructions.

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